Coaxial Cylinders - Rotational MMB

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Two coaxial cylinders of different sizes will have a relative rotation between the two along axes normal to the cylinder axis. The purpose of this short paper is to develop an equation that will allow calculation of the maximum allowable rotation.

1 Preliminaries

Assume a Cartesian coordinate system $\mathcal{C}$. Let cylinders Γ_0, Γ_1 in $\mathcal{C}$ such that the axes of Γ_0, Γ_1 lie along the z-axis. Let Γ_0 be of infinite length with diameter ϕ_M and Γ_1 be of length h with diameter ϕ_L.

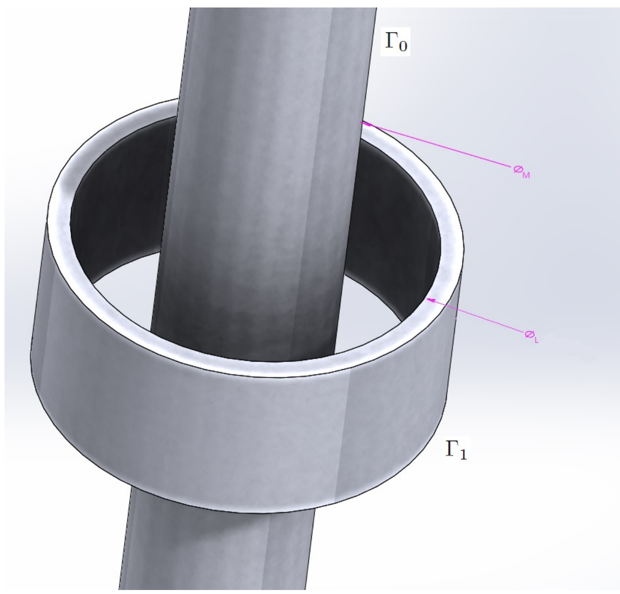

Figure 1: Initial Orientation

2 Geometric Analysis

Assume Γ_1 rotates about an axis normal to the z-axis. The cylinder will be free to rotate until it makes contact with Γ_0. A simple 2D geometric interpretation at the point of contact after rotation about an angle φ is as shown below.

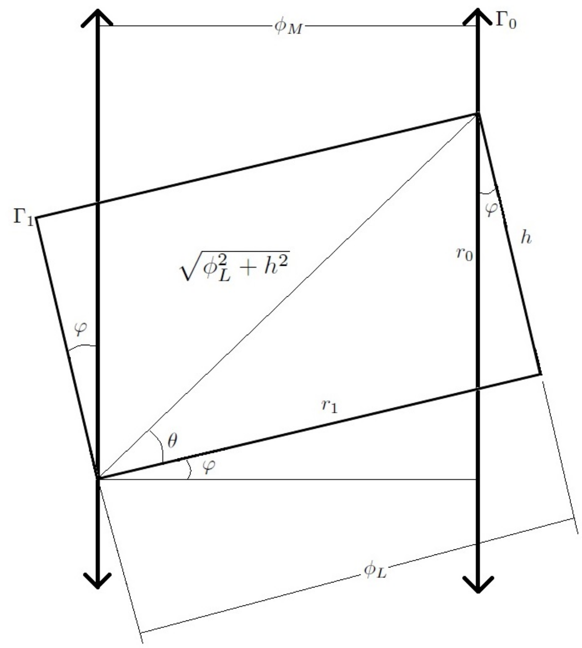

Figure 2: Orientation after rotation

In Figure 2, temporary variables r_0, r_1 have been incorporated for the sake of analysis. Since these quantities are not relevant to the physical situation, we wish to eliminate them. To begin, we can declare a few important relationships. First, we can use a few trigonometric relationships to obtain r_0 in terms of r_1.

$$\cos(\varphi) = \frac{h}{r_0} = \frac{\phi_M}{r_1}$$

$$r_0 = \frac{r_1 h}{\phi_M}$$

Next, using the law of cosines with angle θ yields

$$r_0^2 = r_1^2 + \left(\sqrt{\phi_L^2 + h^2}\right)^2 - 2r_1\sqrt{\phi_L^2 + h^2}\cos(\theta)$$

Examining the interior of Γ_1 shows that

$$\cos(\theta) = \frac{\phi_L}{\sqrt{\phi_L^2 + h^2}}$$

Now, by direct substitution, we have that

$$r_0^2 = r_1^2 + \left(\sqrt{\phi_L^2 + h^2}\right)^2 - 2r_1\phi_L$$

Knowing r_0 in terms of r_1 allows a substitution that will yield

$$\frac{h^2}{\phi_M^2} r_1^2 = r_1^2 + \left(\sqrt{\phi_L^2 + h^2}\right)^2 - 2r_1\phi_L$$

$$0 = \left(1 - \frac{h^2}{\phi_M^2}\right) r_1^2 - 2\phi_L r_1 + \left(\sqrt{\phi_L^2 + h^2}\right)^2$$

We are now left with a quadratic equation in terms of r_1 that is easily solved through the quadratic formula, which states that a quadratic polynomial of the form $Ax^2 + Bx + C$ will have two roots of the form

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

The polynomial equation that has been generated in terms of r_1 will have coefficients $A = \left(1 - \frac{h^2}{\phi_M^2}\right)$, $B = -2\phi_L$, and $C = \left(\sqrt{\phi_L^2 + h^2}\right)^2 = \phi_L^2 + h^2$. Thus, by direct substitution, we have the following roots for r_1

$$r_1 = \frac{2\phi_L \pm \sqrt{4\phi_L^2 - 4\left(1 - \frac{h^2}{\phi_M^2}\right)(\phi_L^2 + h^2)}}{2\left(1 - \frac{h^2}{\phi_M^2}\right)}$$

This expression can be simplified through algebraic manipulation to achieve an appropriate final form for r_1

$$r_1 = \frac{\phi_M^2 \phi_L \pm h\phi_M \sqrt{\phi_L^2 - \phi_M^2 + h^2}}{\phi_M^2 - h^2}$$

Since the goal of this paper is to determine the maximum allowable rotational angle φ, we now look back at a relationship involving φ, namely $\cos(\varphi) = \frac{\phi_M}{r_1}$. Now that r_1 is known, a substitution and simplification can be made such that

$$\cos(\varphi) = \frac{\phi_M^2 - h^2}{\phi_M \phi_L \pm h\sqrt{\phi_L^2 - \phi_M^2 + h^2}}$$

This expression can be evaluated and the inverse cosine can be taken to achieve the maximum allowable rotational angle for the situation as was set up at the beginning. As a final note, the negative sign must be used in the $\pm$. The reasoning for this becomes clear upon evaluating a limiting case. Assume the two cylinders are of the same size such that $\phi_L = \phi_M$. In this situation, absolutely no rotation should be allowed, or $\varphi = 0$ which implies $\cos(\varphi) = 1$. We can make the substitution in the expression for $\cos(\varphi)$ and we obtain

$$\cos(\varphi) = \frac{\phi_M^2 - h^2}{\phi_M \phi_M \pm h\sqrt{\phi_M^2 - \phi_M^2 + h^2}}$$

$$\cos(\varphi) = \frac{\phi_M^2 - h^2}{\phi_M^2 \pm h\sqrt{h^2}}$$

$$\cos(\varphi) = \frac{\phi_M^2 - h^2}{\phi_M^2 \pm h^2}$$

Since we must have that $\cos(\varphi) = 1$, it is clear that we must choose the negative sign. Thus, the expression for the maximum allowable rotation angle is

$$\varphi_{max} = \cos^{-1}\left(\frac{\phi_M^2 - h^2}{\phi_M \phi_L - h\sqrt{\phi_L^2 - \phi_M^2 + h^2}}\right)$$